PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Generalizing Patterns: *Table Tiles*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley
Beta Version

Generalizing Patterns: *Table Tiles*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to identify linear and quadratic relationships in a realistic context: the number of tiles of different types that are needed for a range of square tabletops. In particular, this unit aims to identify and help students who have difficulties with:

- Choosing an appropriate, systematic way to collect and organize data.
- Examining the data and looking for patterns; finding invariance and covariance in the numbers of different types of tile.
- Generalizing using numerical, geometrical or algebraic structure.
- Describing and explaining findings clearly and effectively.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Mathematical Practices* in the *Common Core State Standards for Mathematics*:

7. Look for and make use of structure.
8. Look for and make use of repeated reasoning.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics:*

F-BF: Build a function that models a relationship between two quantities.

INTRODUCTION

The unit is structured in the following way:

- Before the lesson, students attempt the task individually. You then review their work and formulate questions for students to answer in order for them to improve their work.
- At the start of the lesson, students work individually to answer your questions.
- Next, they work collaboratively, in small groups, to produce a better collective solution than those they produced individually. Throughout their work, they justify and explain their decisions to peers.
- In the same small groups, students critique examples of other students' work.
- In a whole-class discussion, students explain and compare the alternative approaches they have seen and used.
- Finally, students work alone again to improve their individual solutions.

MATERIALS REQUIRED

- Each student will need two copies of the worksheet *Table Tiles* and two copies of the *Grid Paper*.
- Each small group of students will need a copy of the *Grid Paper* and a copy of *Sample Responses to Discuss*.
- There are some projectable resources to help you with the whole-class discussions.

TIME NEEDED:

15 minutes before the lesson, a 1-hour lesson, and 10 minutes in a follow-up lesson (or for homework). All timings are approximate. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Table Tiles* (15 minutes)

Have the students do this task in class or for homework a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. Then you will be able to target your help more effectively in the follow-up lesson.

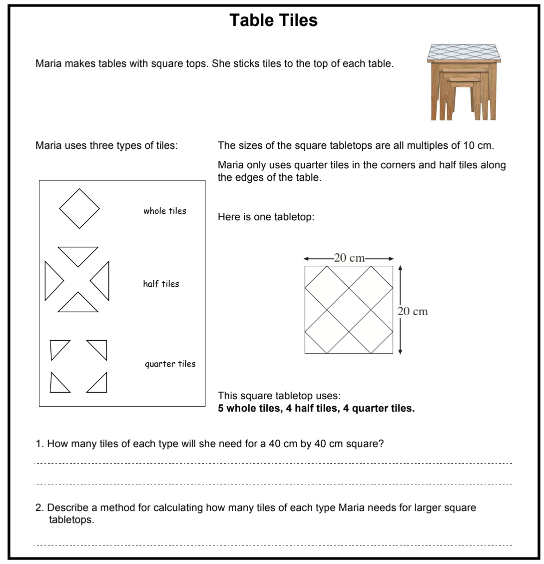


Table Tiles

Maria makes tables with square tops. She sticks tiles to the top of each table.

Maria uses three types of tiles:

whole tiles

half tiles

quarter tiles

The sizes of the square tabletops are all multiples of 10 cm.

Maria only uses quarter tiles in the corners and half tiles along the edges of the table.

Here is one tabletop:

20 cm

20 cm

This square tabletop uses:
5 whole tiles, 4 half tiles, 4 quarter tiles.

1. How many tiles of each type will she need for a 40 cm by 40 cm square?

2. Describe a method for calculating how many tiles of each type Maria needs for larger square tabletops.

Give each student a copy of *Table Tiles* and a copy of the grid paper. Introduce the task briefly and help the class to understand the problem and its context.

Spend 15 minutes on your own, answering these questions.

Show your work on the worksheet and the grid paper.

Don't worry if you can't do everything. There will be a lesson on this material [tomorrow] that will help you improve your work. Your goal is to be able to answer these questions with confidence by the end of that lesson.

It is important that, as far as possible, students answer the questions without assistance.

Students who sit together often produce similar answers so that when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually you ask them to move to different seats. Then, at the beginning of the formative assessment lesson, allow them to return to their usual places. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches. The purpose of this is to forewarn you of the issues that will arise during the lesson, so that you may prepare carefully.

We suggest that you do not score students' work. Research shows that this is counterproductive, as it encourages students to compare scores and distracts their attention from how they may improve their mathematics.

Instead, help students to make further progress by asking questions that focus attention on aspects of their work. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest that you write your own lists of questions, based on your own students' work, using the ideas below. You may choose to write questions on each student's work. If you do not have time to do this, select a few questions that will be of help to the majority of students. These can be written on the board at the beginning of the lesson.

Common issues	Suggested questions and prompts
Student makes unintended assumptions For example: The student has calculated the number of whole tiles required to cover the tabletop, assuming she can split tiles to make quarters and halves as needed. Or: The student uses only quarter tiles to cover the tabletop.	• Imagine you can buy tiles that are ready cut. You don't need to cut them up. How many of each type do you need?
Student makes inaccurate drawing For example: The student divides the whole tabletop into 'units' of a whole tile surrounded by four, quarter tiles. Or: The student draws freehand with a different number of half tiles along each side.	• How would you describe how to draw a 30 cm by 30 cm tabletop?
Student assumes proportionality For example: For Q1 the student writes '10 whole tiles, 8 half tiles, 8 quarter tiles.' The student believes a tabletop with sides twice as long will need twice as many tiles of each type.	• Read the rubric. Where does Maria use quarter tiles? Half tiles? • What happens to tiles in the middle of the diagram if you extend the size?
Unsystematic work For example: The student draws seemingly unconnected examples, such as 10 cm by 10 cm or 40 cm by 40 cm. Or: The student omits some diagrams, drawing tabletops that are 20 cm by 20 cm, and 40 cm by 40 cm, but not 30 cm by 30 cm.	• Which example will you draw next? Why? • What do you notice about the difference between the number of whole tiles in one tabletop and the next? • The sizes of square tabletops are all multiples of 10 cm. Do your diagrams show this?
Student does not generalize For example: The student does not seem to know how to proceed with finding the quadratic expression. Or: The student identifies patterns in the numbers of different types of tiles but does not extend to the general case.	• Can you describe a visual pattern in the number of whole tiles in consecutive diagrams? • How could I find out the number of tiles needed for a larger tabletop, without having to continue the pattern?
Student does not use algebra For example: The student shows awareness of how the number of whole tiles increases with dimensions, but links this to a specific example rather than identifying variables and forming an equation.	• How can you write your answer using mathematical language?

Common issues	Suggested questions and prompts
Student provides a recursive rule not an explicit formula For example: The student provides a way to calculate the number of tiles in a tabletop the next size up from a given size, rather than a general formula for a tabletop of any size; for example, 'next = now + 4', or 'add two more whole tiles than you did last time.'	• Would your method be practical if I wanted to calculate the number of tiles in a 300 cm by 300 cm tabletop?
Student writes incorrect formula For example: The student writes an incorrect formula such as $4x^2 + 4x + 4$ for the number of whole tiles, either using an incorrect algebraic structure or making a recording mistake.	• Does your formula give the correct number of whole tiles in tabletops of different sizes?
Student writes answers without explanation	• How could you explain how you reached your conclusions so that someone in another class understands?
Student correctly identifies constant, linear, and quadratic sequences	• Think of another way of solving the problem. Is this method better or worse than your original one? Explain your answer. • Can you extend your solution to include rectangular tabletops that aren't squares?

SUGGESTED LESSON OUTLINE

Improve individual solutions to the assessment task (10 minutes)

Return your students' work on the *Table Tiles* problem. Ask students to re-read both the *Table Tiles* problem and their solutions. If you have not added questions to students' work, write a short list of your most common questions on the board. Students can then select a few questions appropriate to their own work and begin answering them.

Recall what we were working on previously. What was the task?

Draw students' attention to the questions you have written.

I have read your solutions and have some questions about your work.

I would like you to work on your own for 10 minutes to answer my questions.

Collaborative small-group work (15 minutes)

Organize the students into small groups of two or three. In trials, teachers found keeping groups small helped more students play an active role. Give each group a new sheet of grid paper.

Students should now work together to produce a joint solution.

Put your solutions aside until later in the lesson. I want you to work in groups now.

Your task is to work together to produce a solution that is better than your individual solutions.

You have two tasks during small-group work: to note different student approaches to the task, and to support student problem solving.

Note different student approaches to the task

Notice how students work on finding the quadratic function for the number of whole tiles. Notice also whether and when students introduce algebra. If they do use algebra, note the different formulations of the functions they produce, including incorrect versions, for use in whole-class discussion. You can use this information to focus the whole-class plenary discussion towards the end of the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to this task. Instead, ask questions to help students clarify their thinking. If several students in the class are struggling with the same issue, you could write a relevant question on the board. You might also ask a student who has performed well on one part of the task to help a student struggling with that part of the task.

The following questions and prompts have been found most helpful in trials:

What information have you been given?

What do you need to find out?

What changes between these diagrams? What stays the same?

What if I wanted to know the thousandth example?

How will you write down your pattern?

Why do you think your conjecture might be true?

In trials, teachers expressed surprise at the degree of difficulty some students experienced in drawing the tabletops. If this issue arises in your class, help the student to focus his or her attention on different parts of the tabletop, how they align with the grid, and then get them to draw the whole diagram from those pieces.

Where are the half tiles? Whole tiles?

How do the whole and half tiles fit together?

You may find that some students do not work systematically when drawing tabletops and organizing their data.

What sizes of tabletop might Maria make? Which ones is it useful for you to draw? Why?

What can you do to organize your data?

If students have found formulas, focus their attention on improving explanations, exploring alternative methods, and showing the equivalence of different equations.

How can you be sure your explanation works in all cases?

Ask another group if your argument makes sense.

Which is the formula you prefer? Why?

Show me that these two expressions are equivalent.

Students may justify their formulas by drawing another example to see if the generalization fits a new case, reasoning inductively. Some stronger explanations are shown in the *Sample Responses to Discuss*.

Collaborative analysis of *Sample Responses to Discuss* (15 minutes)

Give each small group of students a copy of the *Sample Responses to Discuss*. These are three of the common problem-solving approaches taken by students in trials. Display the following questions on the board or project slide P-1 *Student Responses to Discuss*.

Describe the problem solving approach the student used.

You might, for example:

- *Describe the way the student has colored the pattern of tiles.*
- *Describe what the student did to calculate a sequence of numbers.*

Explain what the student could do to complete his or her solution.

This analysis task will give students an opportunity to evaluate a variety of alternative approaches to the task, without providing a complete solution strategy.

During small-group work, support student thinking as before. Also, check to see which of the explanations students find more difficult to understand. Identify one or two of these approaches to discuss in the plenary discussion. Note similarities and differences between the sample approaches and those the students took in small-group work.

Whole-class discussion comparing different approaches (20 minutes)

Organize a whole-class discussion to consider different approaches to the task. The intention is for you to focus on getting students to understand the methods of working out the answers, rather than either numerical or algebraic solutions. Focus your discussion on parts of the two small-group tasks students found difficult. You may find it helpful to display slide P-2, *Grid Paper*, or slide P-3 *Tabletops*.

Let's stop and talk about different approaches.

Ask the students to compare the different solution methods.

Which approach did you like best? Why?

Which approach did you find it most difficult to understand?

Sami, your group used that method. Can you explain that for us?

Which method would work best for the thousandth tabletop?

Below, we have given details of some discussions that emerged in trial lessons.

Some students found the work on quadratic expressions very difficult. If your students have this problem, you might focus on Gianna's method from the *Sample Responses to Discuss* (slide P4).

Describe Gianna's pattern in the whole tiles in the 30 cm by 30 cm tabletop.

How would you describe her pattern in the next size tabletop?

Using Gianna's pattern, how many whole tiles would there be in any tabletop?

If students have found different algebraic formulations for the number of half and whole tiles, it might help to write a variety of their expressions on the board. Ask students to link different variables and manipulate algebraic expressions to identify errors and show equivalences:

Which of these formulas would you use to find the number of half tiles?

Which are quadratic?

Are there any formulas that are equivalent?

Review individual solutions to the assessment task (10 minutes)

If you are running out of time, you could schedule this activity for the next lesson or for homework.

Make sure students have their original individual work on the *Table Tiles* task to hand. Give them a fresh, blank copy of the *Table Tiles* task sheet and of the *Grid Paper*.

Read through your original responses and think about what you have learned this lesson.

Using what you have learned, try to improve your work.

If a student is satisfied with his or her solution, ask the student to try a different approach to the problem and to compare the approach already used.

SOLUTIONS

Size of tabletop (cm)	10 × 10	20 × 20	30 × 30	40 × 40	50 × 50	60 × 60
Number of quarter tiles	4	4	4	4	4	4
Number of half tiles	0	4	8	12	16	20
Number of whole tiles	1	5	13	25	41	61

For a tabletop of side length x, $n = \dfrac{x}{10}$ is the number of tile diagonal widths in the side length.

The number of quarter tiles is always 4. The number of half tiles is $4(n-1) = 4\left(\dfrac{x}{10} - 1\right)$.

The number of whole tiles is $n^2 + (n-1)^2 = 2n^2 - 2n + 1 = 2\left(\dfrac{x}{10}\right)^2 - 2\left(\dfrac{x}{10} - 1\right)^2 = \left(\dfrac{x}{10}\right)^2 + \left(\dfrac{x}{10} - 1\right)^2$.

Analysis of Student Responses to Discuss

Leon's method

Leon drew three diagrams showing tabletops with systematically increasing side lengths. He colored the diagrams to pick out rows of whole squares parallel to the diagonal of the tabletop.

Width of tabletop	10 cm	20 cm	30 cm
Number of whole tiles	1	1 + 3 + 1	1 + 3 + 5 + 3 + 1

Leon wrote the sum of the numbers to show the total number of whole tiles. He did this in an organized way. He predicted the number of whole tiles in the next diagram accurately.

Leon could check the next diagram to see if his conjecture is correct, but this would not be a proof. He could use his method to predict the number of whole tiles in the next diagram for any diagram he has drawn, but that method would not enable him to calculate the number of whole tiles for tables of any size. To do that, he might attempt to articulate the relationship between the number of whole tiles across the diagonal and the number of odd numbers to sum (assuming that students do not recognize that $1 + 3 + 5 + \ldots + 2n - 1 = n^2$).

Gianna's method

Gianna shaded alternate horizontal rows of squares. In the first two diagrams she wrote numbers in the whole squares. These record the number of whole squares in each horizontal row of that tabletop.

For the diagram of the 30 cm tabletop this gives:

In the first row:	3		3		3
In the second row:		2		2	
In the third row:	3		3		3
In the fourth row:		2		2	
In the fifth row:	3		3		3

Gianna picked out the pattern as 3 rows of 3 whole tiles, and 2 rows of 2 whole tiles, to find the total number of whole tiles is $3 \times 3 + 2 \times 2$.

Gianna could then generalize to show that in the n^{th} tabletop there would be n rows of n whole tiles, and $n-1$ rows of $n-1$ whole tiles, in total $n^2+(n-1)^2$ whole tiles.

Ava's method

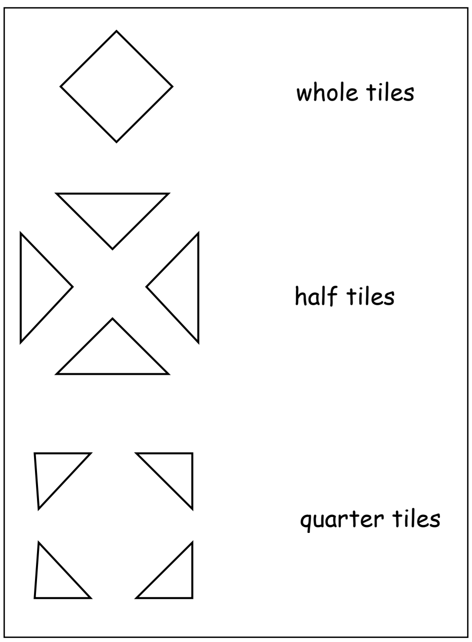
Ava drew one diagram showing how the length of a 40 cm tableside is made of two 5 cm edges of quarter tiles and three 10cm half tiles, and another similar diagram for the 50 cm table. Ava systematically organized data in a table, although it is not clear where the data came from. She found differences and second differences between numbers of tiles in tables of increasing side length. She did not show a way of calculating the number of tiles of various types given an arbitrary table size, and she did not use algebra.

Ava could next use her table of data to derive formulas for the number of tiles in any table. She might do this by using the first and second differences with the standard quadratic sequence algorithm.

Table Tiles

Maria makes tables with square tops. She sticks tiles to the top of each table.

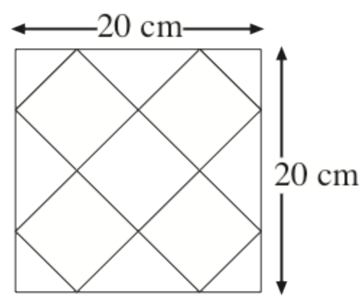
Maria uses three types of tiles:



The sizes of the square tabletops are all multiples of 10 cm.

Maria only uses quarter tiles in the corners and half tiles along the edges of the table.

Here is one tabletop:



This square tabletop uses:
5 whole tiles, 4 half tiles, 4 quarter tiles.

1. How many tiles of each type will she need for a 40 cm by 40 cm square?

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2. Describe a method for calculating how many tiles of each type Maria needs for larger square tabletops.

Grid Paper

Sample Responses to Discuss

Here is some work on *Table Tiles* from three students in another class: Leon, Ava and Gianna.

For each piece of work:

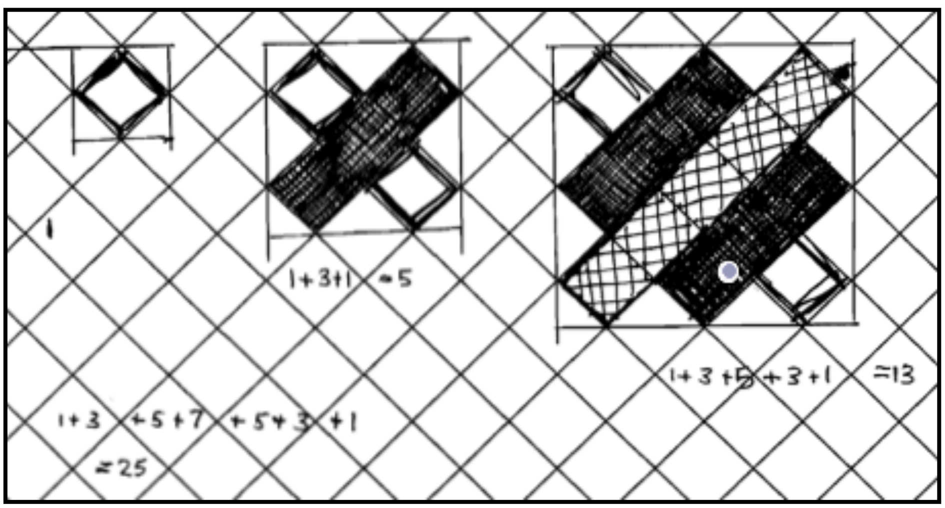
1. Describe the problem solving approach the student used.

   For example, you might:

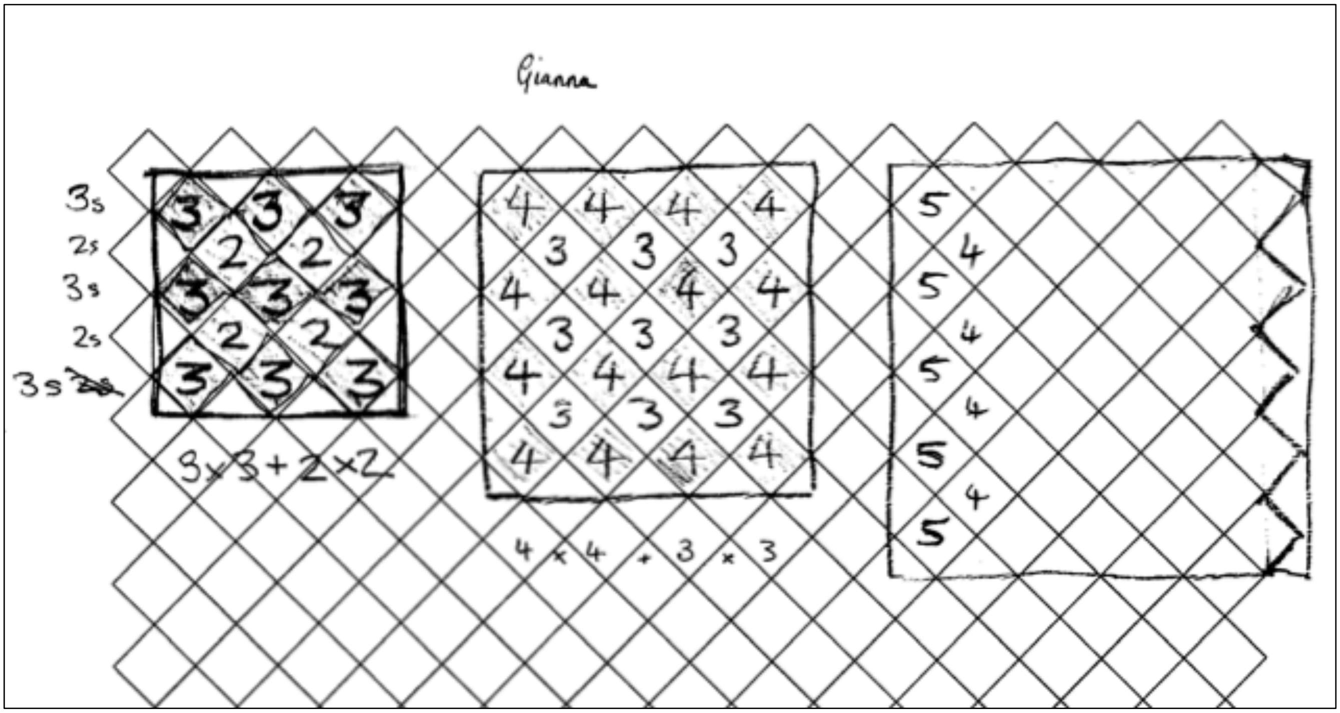
   - Describe the way the student has colored the pattern of tiles.
   - Describe what the student did to calculate a sequence of numbers.

2. Explain what the student needs to do to complete his or her solution.

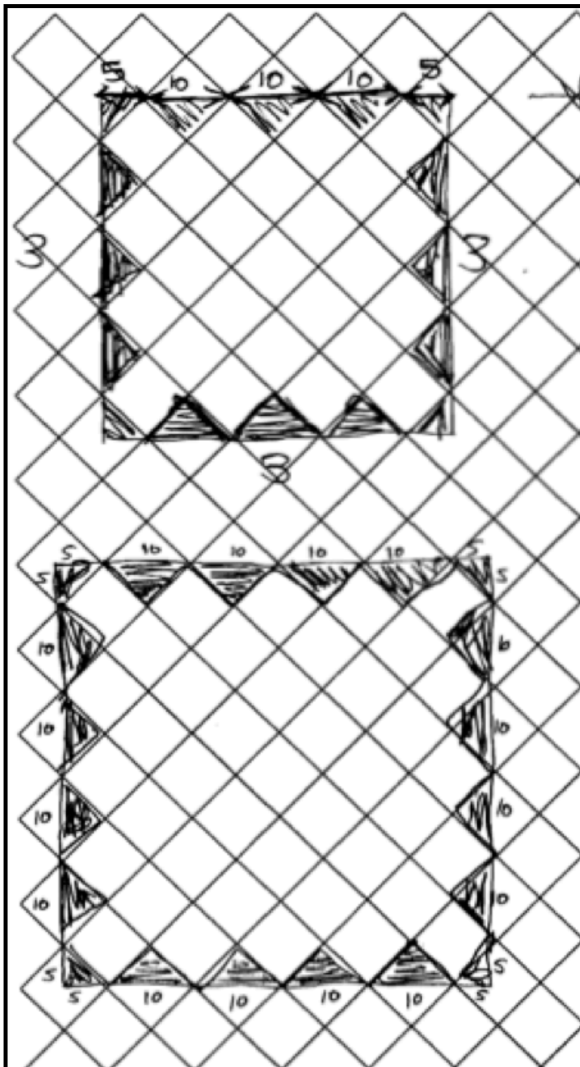
Leon's method



Gianna's method



Ava's method



Side length	10cm	20cm	30cm	40cm	50cm
Quarters	4	4	4	4	4
Halfs	0	4	8	12	16
Wholes	1	5	13	25	41

4 8 12 16

+4 +4 +4

Student Responses to Discuss

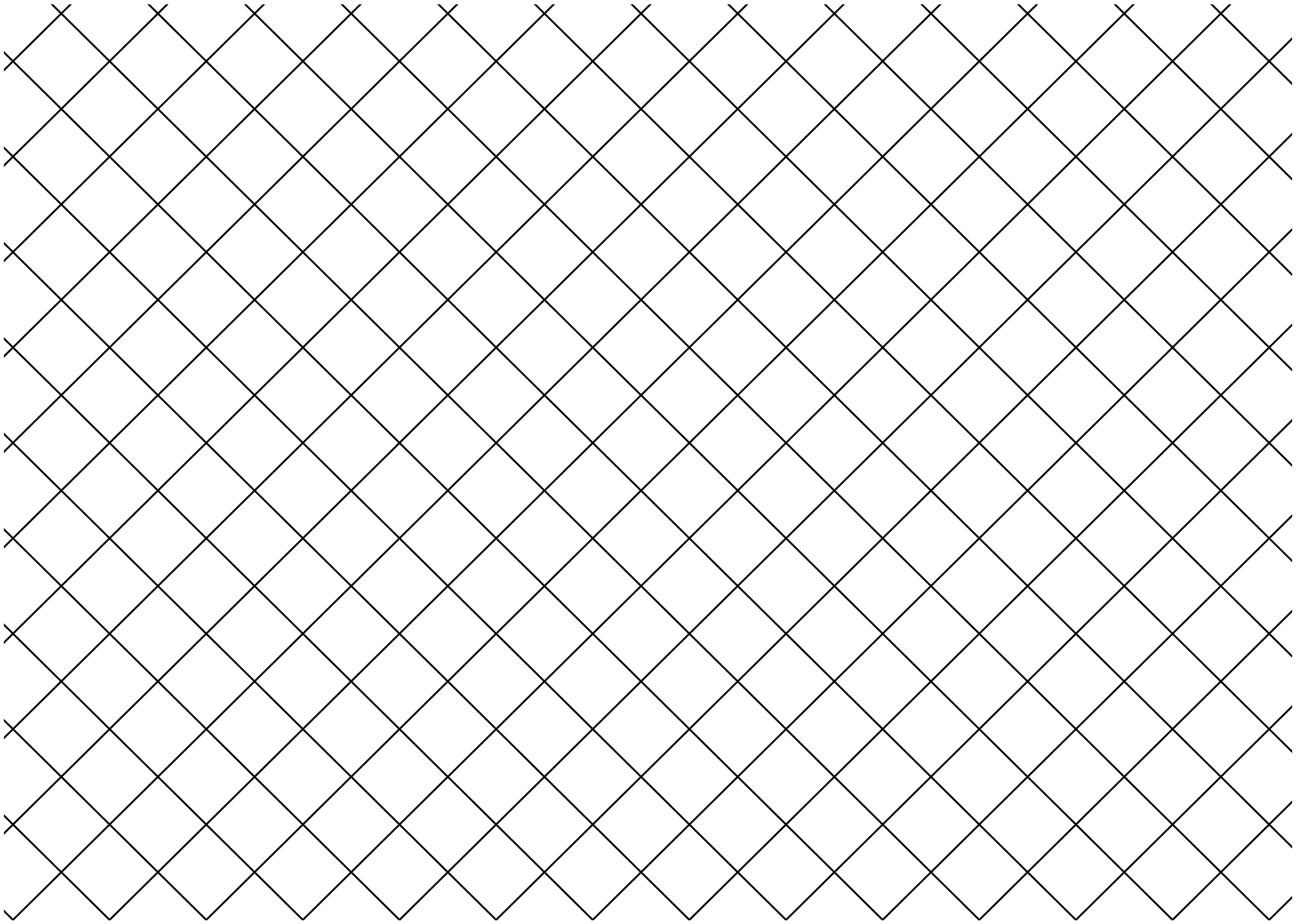
1. Describe the problem solving approach the student used.

   You might, for example:

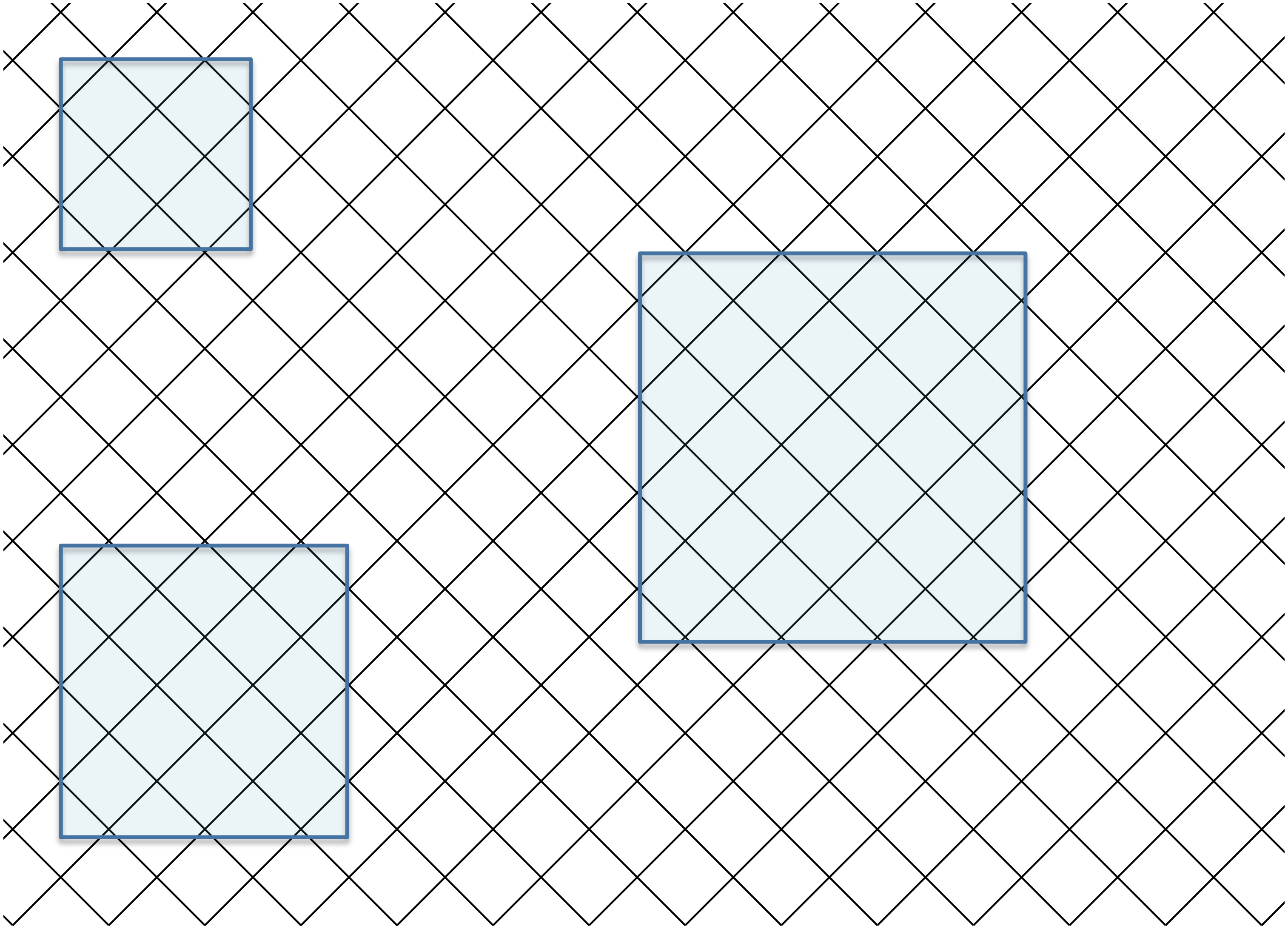
   - Describe the way the student has colored the pattern of tiles.
   - Describe what the student did to calculate a sequence of numbers.

2. Explain what the student needs to do to complete his or her solution.

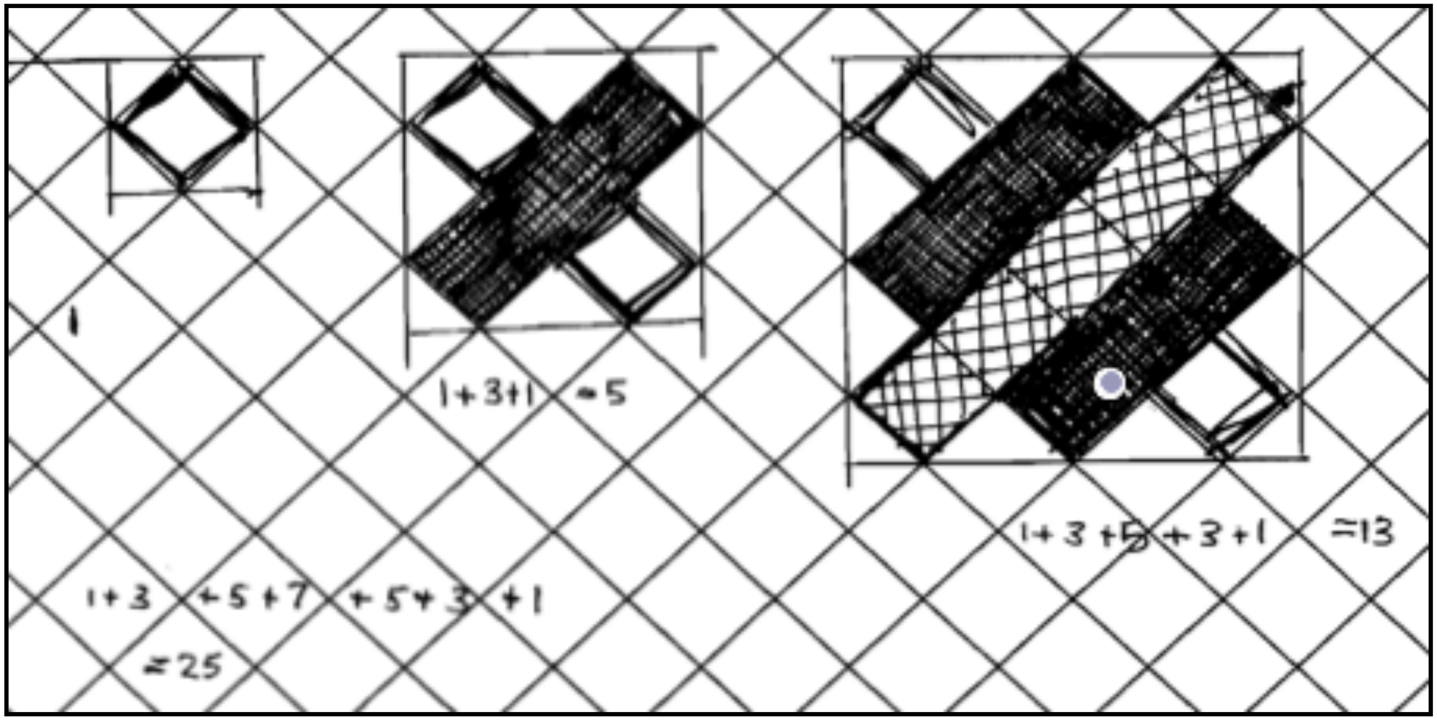
Grid Paper



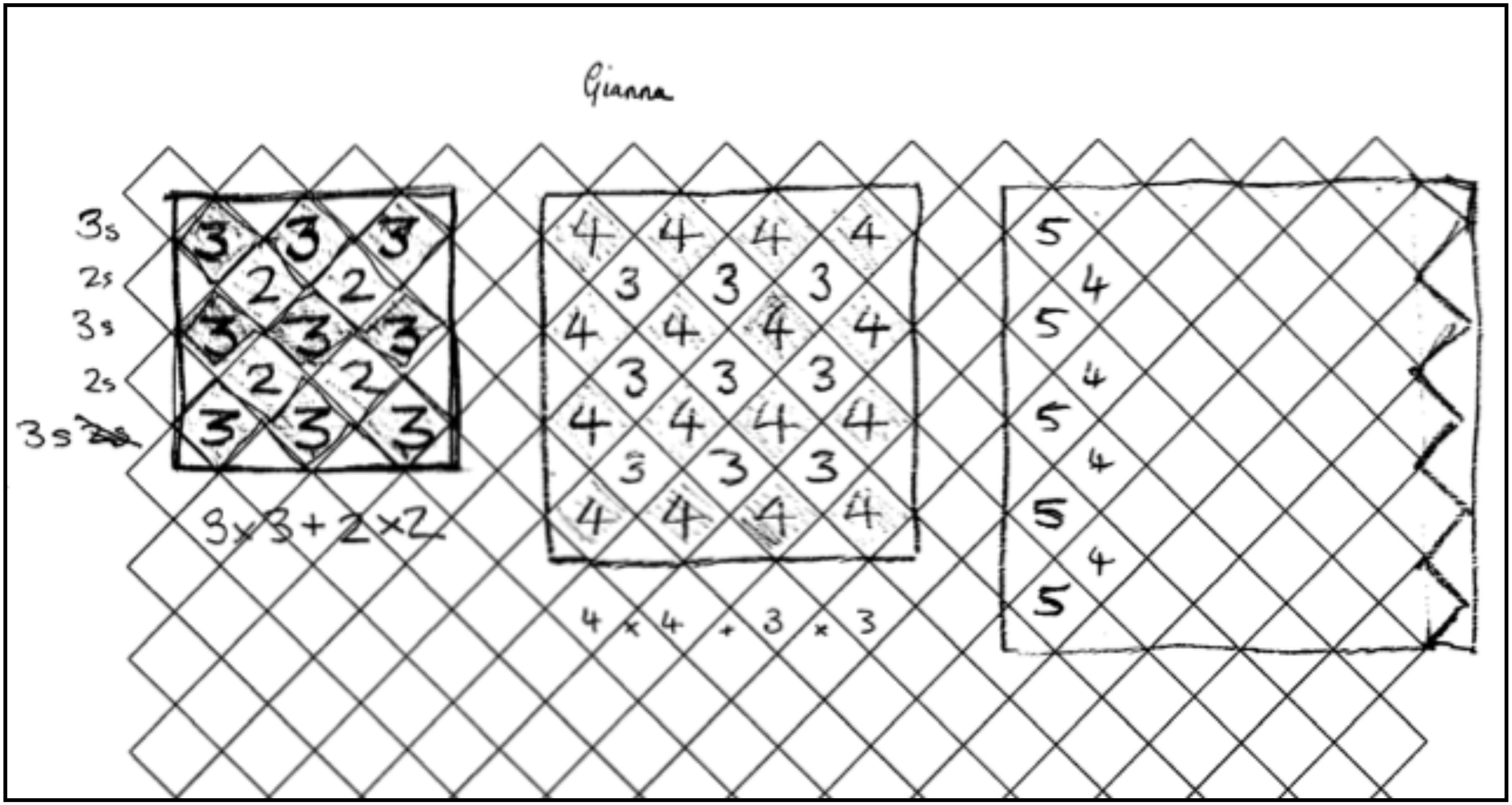
Tabletops



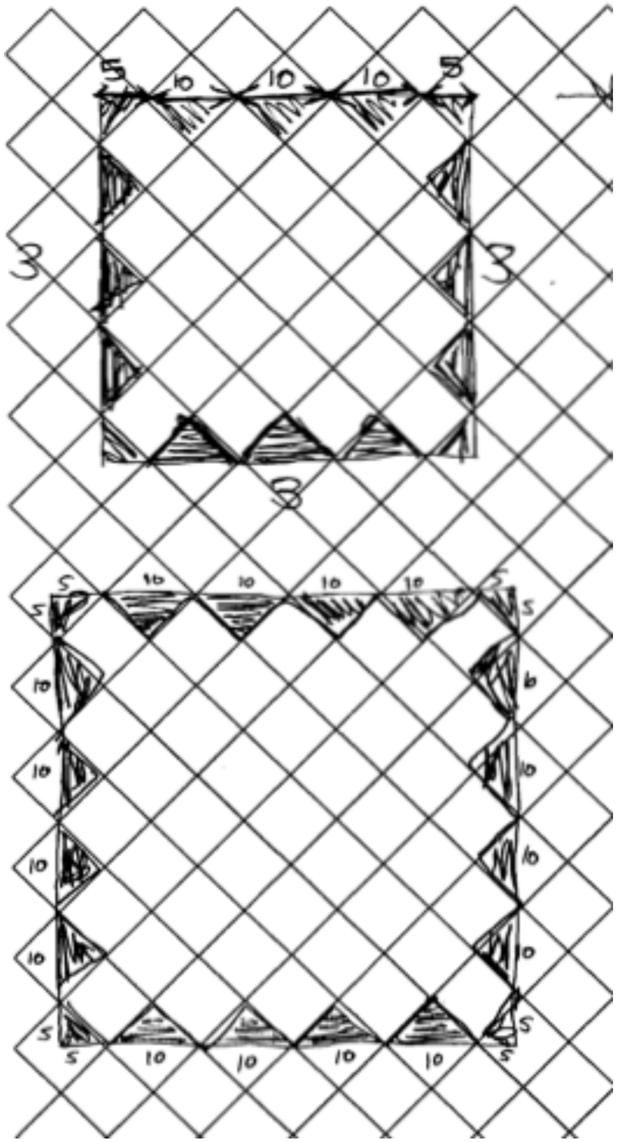
Leon's method



Gianna’s method



Ava’s method



Side length	10cm	20cm	30cm	40cm	50cm
Quarters	4	4	4	4	4
Half	0	4	8	12	16
Wholes	1	5	13	25	41

4 8 12 16

+4 +4 +4

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
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It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

We are grateful to the many teachers, in the UK and the US, who trialed earlier versions
of these materials in their classrooms, to their students, and to
Judith Mills, Carol Hill, and Alvaro Villanueva who contributed to the design.

This development would not have been possible without the support of
Bill & Melinda Gates Foundation
We are particularly grateful to
Carina Wong, Melissa Chabran, and Jamie McKee